Crossian Resonance

A Multidisciplinary Research Journal **ISSN 0976-5417**

(A refereed Biannual Published in June and December)

Vol. 15, No.2, December 2024

https://research.holycrossngl.edu.in/

Signed Product and Signed Total Product Domination Number of Some Standard Graphs

ShekinahHenry B.

Department of Mathematics, Holy Cross College (Autonomous), Nagercoil - 629 004

Affiliated to Manonmaniam Sundaranar University, Tirunelveli - 627 012

ABSTRACT

This paper explores the signed product domination number and signed total product domination number of various standard graphs. The paper provides explicit computations and proofs for these domination numbers across different graph classes including paths, cycles, complete graphs, and bipartite graphs.

Keywords: *Signed Product Domination Number, Signed Total Product Domination Number*

1. Introduction

Graph theory is a vital branch in combinatorial mathematics, offering critical insights into the structural properties of networks, paths, and cycles. Domination in graphs, particularly through signed and product functions, has emerged as a significant area of study due to its applications in network theory, optimization, and algorithm design. This paper focuses on two specialized domination parameters: the signed product domination number $\gamma_{s^*}(G)$ and the signed total product domination number $\gamma_{(st)^*}(G)$. By a graph we mean a finite, undirected, connected graph without loops or multiple edges. Terms not defined here are used in the sense of [1].

Dunbar introduced the concept of signed dominating function. Let $G = (V, E)$ be a graph. A function $f: V(G) \to \{-1,1\}$ is a signed dominating function if $f(N[v]) = \sum_{u \epsilon N[v]} f(u) \geq 1$ for all $v \epsilon V(G)$. For a real valued function f the weight of f is $w(f) = \sum_{v \in V} f(v)$ [2]. The minimum weight $w(f) = f(V(G)) = \sum_{x \epsilon V(G)} f(x)$ taken over all signed dominating functions is the signed domination number of G. It is denoted by $\gamma_s(G)$. A function $f: V(G) \to \{-1,1\}$ is a signed total dominating function if $f((V)) = \sum_{u \epsilon N(v)} f(u) \geq 1$ for all $v \epsilon V(G)$ [3]. The minimum weight $w(f) = f(V(G)) = \sum_{x \epsilon V(G)} f(x)$ taken over all signed total dominating functions is the signed domination number of G. It is denoted by $\gamma_{st}(G)$. Signed domination number of complete multipartite graph and some standard graphs are found in [4, 5]. In this paper signed product dominating function and signed total product dominating function are introduced. Also signed product domination number and signed total product domination number of paths, cycle, complete and bipartite graphs are found.

2. Main Results

Definition 2.1. Let $G = (V, E)$ be a graph. A function $f : V(G) \to \{-1, 1\}$ is a signed product dominating function if f is a signed dominating function and $\prod f(N[v]) = \prod_{u\in N[v]} f(u) = 1$ for all $v \in V(G)$.

Definition 2.2. The minimum weight $w(f) = f\big(V(G)\big) = \sum_{x\in V(G)} f(x)$ taken over all signed product dominating functions is the signed product domination number of G. It is denoted by $\gamma_{s^*}(G)$.

Definition 2.3. Let $G = (V, E)$ be a graph. A function $f : V(G) \to \{-1, 1\}$ is a signed total product dominating function if f is a signed total dominating function and $\prod f(N(v)) = \prod_{u\in N(v)} f(u) = 1$ for all $v \in V(G)$.

Definition 2.4. The minimum weight $w(f) = f\big(V(G)\big) = \sum_{x\in V(G)} f(x)$ taken over all signed total product dominating functions is the signed total product domi- nation number of G. It is denoted by $\gamma_{(st)^*}(G)$.

Theorem 2.5. For a path $P_n, \gamma_{s^*}(P_n) = n$.

Proof. *Let* $f : V(P_n) \to \{-1, 1\}$ be a signed product dominating function.
Then $f(v) = +1 \forall v \in V(P_n)$ otherwise $\prod_{u\in N[v]} f(u) = -1$ for some $v \in V(P_n)$.
Hence $\gamma_{s^*}(P_n) = n$.

Theorem 2.6. For a cycle C_n, $\gamma_{s^*}(C_n) = n$.

Proof. Let $f : V(C_n) \to \{-1, 1\}$ be a signed product dominating function.
Then $f(v) = +1 \forall v \in V(C_n)$ otherwise $\prod_{u\in N[v]} f(u) = -1$for some v $\in V(C_n)$, since every vertex of C_n has degree 2.
Hence $\gamma_{s^*}(C_n) = n$.

Theorem 2.7. Let G be a complete graph. Then the signed product domination number of G is

$$\gamma_{s^*}(G) = \begin{cases} 1 \ if \ n \ is \ odd \ and \ \frac{n-1}{2} \ even \\ 2 \ if \ n \ is \ even \ and \ \frac{n}{2} - 1 \ even \\ 3 \ if \ n \ is \ odd \ and \ \frac{n-1}{2} \ odd \\ 4 \ if \ n \ is \ even \ and \ \frac{n}{2} - 1 \ odd \end{cases}$$

Proof. Let $f : V(G) \to \{-1, 1\}$ be a signed product dominating function with $w(f) = \gamma_{s^*}(G)$. Then

$\sum_{v\in N[u]} f(v) \geq 1$ and $\prod_{v\in N[u]} f(v) = 1$.

Let $A = \{v \in V(G)|f(v) = 1\}$, $B = \{v \in V(G)|f(v) = -1\}$. Since G is a complete graph, $\sum_{v\in N[u]} f(v) = f(V) = w(f)$.

$$|A| - |B| \geq 1$$

$$|A| \geq 1 + |B|$$

$$|A| + |B| = n$$

This implies that $n - |B| \geq 1 + |B|$

$$|B| \leq \frac{n-1}{2}.$$

Case 1: n is odd and $\frac{n-1}{2}$ is even

Since $\frac{n-1}{2}$ is even, $\prod_{v\in N[u]} f(v) = 1$ if $|B| = \frac{n-1}{2}$

Hence $(f) = \frac{n+1}{2} - \frac{n-1}{2} = 1$.

Case 2: n is odd and $\frac{n-1}{2}$ is odd.

Since $\frac{n-1}{2}$ is odd and $|B| \leq \frac{n-1}{2}$, $\prod_{v\in N[u]} f(v) = -1$ if $|B| = \frac{n-1}{2}$

Hence $|B| = \frac{n+1}{2} - 1 = \frac{n-3}{2}$.

This implies $w(f) = \frac{n+3}{2} - \frac{n-3}{2} = 3$

Case 3: n even and $\frac{n}{2} - 1$ is even.

We have $|B| \leq \frac{n-1}{2}$.

Also, since n is even $\frac{n-1}{2}$ is not an integer.

Hence $|B| \leq \left\lfloor\frac{n-1}{2}\right\rfloor = \frac{n}{2} - 1$.

Now if $|B| = \frac{n}{2} - 1$, then $\prod_{v\in N[u]} f(v) = 1$

Thus $w(f) = \frac{n+2}{2} - \frac{n-2}{2} = 2$.

Case 4: n even and $\frac{n}{2} - 1$ is odd.

Since n is even $\frac{n-1}{2}$ is not an integer.

Hence $|B| \leq \left\lfloor\frac{n-1}{2}\right\rfloor = \frac{n}{2} - 1$.

Now if $|B| = n/2 - 1$, then $\prod_{v\in N[u]} f(v) = -1$

Thus $w(f) = \frac{n+2}{2} - \frac{n-2}{2} = 2$.

which implies that f is not a signed product dominating function.

Thus $|B| = \frac{n}{2} - 1 - 1 = \frac{n-4}{2}$.

Therefore $w(f) = \frac{n+4}{2} - \frac{n-4}{2} = 4$

Theorem 2.8. For a complete bipartite graph $K_{p,q}$, $p \leq q, \gamma_{s^*}(K_{p,q}) = p + q$.

Proof. Let V_1 and V_2 be the partite sets with $V_1 = \{v_{1,j} \mid 1 \leq j \leq p\}, V_2 = \{v_{2,j} \mid 1 \leq j \leq q\}$.

Let f be a signed product dominating function with minimum weight.

Then $f\left[N\left[v_{i,j}\right]\right] = f\left(v_{1,j}\right) + f\left[V/V_i\right] \geq 1, i = 1, 1 \leq j \leq p,$

$i = 2, 1 \leq j \leq q$ and

$$\prod f\left(N\left[v_{i,j}\right]\right) = \prod_{u_{i,j} \in N[v_{i,j}]} f\left(u_{i,j}\right) = f\left(v_{i,j}\right) \times \prod f\left[V/V_i\right] = 1,$$

$$i = 1,2, 1 \leq j \leq p \leq q.$$

Thus, if $f(v_{i,j}) = -1$, then $\prod f[V/V_i] = -1$.

Also, if $f(v_{i,j}) = 1$, then $\prod f[V/V_i] = 1$.

Let $A = \{v \in V(K_{p,q}) | f(v) = 1\}, B = \{v \in V(K_{p,q}) | f(v) = -1\}$.

Claim: Either $V_i \subseteq B$ or $v_{i,j} \notin B$ for all $v_{i,j} \in V_i$.

Suppose V_i is not contained in B.

Then there is a vertex $v_{i,j} \in V_i$, such that $f(v_{i,j}) = 1$

Thus $\prod f[V/V_i] = 1$

Assume that $v_{i,j} \in B$ for some $v_{i,j} \in V_i$, then $f(v_{i,j}) = -1$.

$$\prod f\left(N\left[v_{i,j}\right]\right) = f\left(v_{i,j}\right) \times \prod f\left[V/V_i\right] = -1,$$

This implies f is not a signed product dominating function.

Hence $v_{i,j} \notin B$ for all $v_{i,j} \in V_i$

Claim: $B = \emptyset$

Suppose $B \neq \emptyset$, then there is a vertex $v_{i,j} \in B$. Then by claim 1, $V_i \subseteq B$ for at least one i, i = 1,2. Thus for $v_{k,j} \in V_k$, $k \neq i, k = 1, 2$,

$f(N[v_{k,j}]) = f(v_{k,j}) + f[V/V_k] < 1$, since $f[V/V_k] = f[V_i] < 0$.

This implies f is not a signed product dominating function, which is a contradiction.

Thus $B = \emptyset$.

Hence $\gamma_{s^*}\left(K_{p,q}\right) = w(f) = |A| - |B| = p + q$.

Theorem 2.9. For a complete tripartite graph K_{n_1,n_2,n_3}, the signed product domination number is $\gamma_{s^*}(K_{n_1,n_2,n_3}) = n_1 + n_2 + n_3$.

Proof. Let $G = K_{n_1,n_2,n_3}$

$$V(G) = \{v_{i,j} | 1 \leq j \leq n_i\}, i = 1, 2, 3.$$

Also let f be a signed product dominating function with $w(f) = \gamma_{s^*}(G)$. Then for $1 \le i \le 3, 1 \le j \le n_i$,

$$f\,(N_G[v_{i,j}]) = f\,(v_{i,j}) + f\,[V/Vi] \ge 1$$

and

$\prod f\,(N_G[v_{i,j}]) = f\,(v_{i,j}) \times \prod f\,[V/Vi] = 1.$

Claim 1: Either $V_i \subseteq B$ or $v_{i,j} \notin B$ for all $v_{i,j} \in V_i$.

Suppose V_i is not contained in B.

Then there is a vertex $v_{i,j} \in V_i$ such that $f\,(v_{i,j}) = 1$

Thus $\prod f\,[V/\,V_i] = 1$, since $\prod f\,(N_G[v_{i,j}]) = 1$.

If there is another element $v_{i,k} \in B$ for some $v_{i,k} \in V_i$, then $f\,(v_{i,j}) = -1$

This implies $\prod f(N[v_{i,j}]) = f\,(v_{i,j}) \times \prod f\,[V/\,V_i] = -1,$

This implies f is not a signed product dominating function.

Thus $v_{i,j} \notin B$ for all$v_{i,j} \notin V_i$.

Claim 2:

If $V_i \subseteq B$, there exist $a\,k, k = 1,2,3$ and $i \neq k$ such that $V_k \subseteq B$.

Let $V_i \subseteq B$,

Then by claim 1 $f\,(v_{i,j}) = -1$ for all $v_{i,j} \in V_i$. Since $\prod f\,(N_G[v_{i,j}]) = 1$,

$\prod f\,[V/\,V_i] = -1$

Then there exists at least one vertex $v_{k,j} \in V_k$ such that $f\,(v_{k,j}) = -1$.

Again, by claim 1 $f\,(v_{k,j}) = -1$ for all $v_{k,j} \in V_k$.

Hence $V_k \subseteq B$.

Claim 3: $B = \emptyset$

Suppose $B \neq \emptyset$, then there is a vertex $v_{i,j} \in B$. Then by claim 1and 2, $V_i \subseteq B$ for at least two $i's, i = 1,2$.

Consider the partite set k for which V_k not contained in B.

Thus for $v_{k,j} \in V_k,\ k \neq i, k = 1,2,3$

$f\,(N\,[v_{k,j}]) = f\,(v_{k,j}) + f\,[V/V_k] < 1$, since $f\,[V/V_k] = f\,[V_i] < 0$.

This implies f is not a signed product dominating function, which is a contradiction. Thus $B = \emptyset$.

Hence $\gamma_{s^*}(K_{n_1,n_2,n_3}) = n_1 + n_2 + n_3$.

Theorem 2.10. For a path $P_n, \gamma_{(st)^*}(P_n) = n$.

Proof. Let $f : V(P_n) \to \{-1,1\}$ be a signed total product dominating function. Then $f(v) = +1 \forall v \in V(P_n)$ otherwise $\prod_{u\in N(v)} f(u) = -1$ for some $v \in V(P_n)$.

Hence $\gamma_{(st)^*}(P_n) = n$.

Theorem 2.11. For a cycle C_n, $\gamma_{(st)^*}(C_n) = n$.

Proof. Let $f : V(C_n) \to \{-1,1\}$ be a signed total product dominating function. Then $f(v) = +1 \forall v \in V(C_n)$ otherwise $\prod_{u\in(v)} f(u) = -1$for some v $\in V(C_n)$, since every vertex of C_n has degree 2.

Hence $\gamma_{(st)^*}(C_n) = n$.

***Theorem* 2.12.** For a complete graph G of order n, $\gamma_{(st)^*}(G) = n$.

Proof. Let $f : V(G) \to \{-1,1\}$ be a signed total product dominating function with $w(f) = \gamma_{(st)^*}(G)$.Then

$\sum_{v\in N(u)} f(v) \geq 1$ and $\prod_{v\in N(u)} f(v) = 1$.

Let A $= \{v \in V(G) | f(v) = 1\}, B = \{v \in V(G) | f(v) = -1\}$.

Claim: $B = \emptyset$

Suppose $B \neq \emptyset$

Then there exist at least one $v \in V(G)$ such that $f(v) = -1$.

Since $\prod_{v\in N(u)} f(v) = 1$., for all $u \in V(G)$ there must be another vertex $u \in V(G)$, such that $f(u) = -1$.

Then $\prod_{v\in N(u)} f(v) = -1$. where $u \in B$, which is a contradiction. Thus $B = \emptyset$.

This implies $V(G) \subseteq A$.

Hence $\gamma_{(st)^*}(G) = n$.

Theorem 2.13. Let $K_{p,q}$,$p \leq q$ be a complete bipartite graph with p even and $\frac{p}{2} - 1$ even, then

$$\gamma_{(st)^*}(K_{p,q}) = \begin{cases} 3 \ if \ q \ is \ odd \ and \ \frac{q-1}{2} \ even \\ 4 \ if \ q \ is \ even \ and \ \frac{q}{2} - 1 \ even \\ 5 \ if \ q \ is \ odd \ and \ \frac{q-1}{2} \ odd \\ 6 \ if \ q \ is \ even \ and \ \frac{q}{2} - 1 \ odd \end{cases}.$$

Proof. Let P be a partite set with p vertices and Q be a partite set with q vertices. Let $f: (V(K_{p,q})) \to \{-1,1\}$ be a signed total product dominating function with minimum weight.

And $A = \{v \in V(K_{p,q}) | f(v) = 1\}$and

$B = \{v \in V(K_{p,q}) | f(v) = -1\}$.

Also let $P^+ = \{v \in V(P) | f(v) = 1\}$, $P^- = \{v \in V(P) | f(v) = -1\}$

$Q^+ = \{v \in V(Q) | f(v) = 1\}$ and $Q^- = \{v \in V(Q) | f(v) = -1\}$.

Now $| P^-| \leq \frac{p}{2} - 1$, since p is even.

Also, since $\frac{p}{2} - 1$ is even $\prod_{v \in N(u)} f(v) = 1$for $u \in Q$.

Hence $| P^-| = \frac{p}{2} - 1$ and $| P^+| = \frac{p}{2} + 1$.

Case 1: q is odd and $\frac{q-1}{2}$ even.

Since q is odd$| Q^-| \leq \frac{q-1}{2}$.

Also, since $\frac{q-1}{2}$ is even $\prod_{u \in N(v)} f(u) = 1$for $v \in P$.

Now
$$|B| = |P^-| + |Q^-|$$
$$= \frac{p}{2} - 1 + \frac{q-1}{2} = \frac{p+q-3}{2}$$
$$|A| = p + q - \left[\frac{p+q-3}{2}\right] = \frac{p+q+3}{2}.$$
$$\gamma_{(st)^*}(K_{p,q}) = w(f) = |A| + |B| = 3.$$

Case 2: q is even and $\frac{q}{2} - 1$ even.

Since q is even$| Q^-| \leq \frac{q}{2} - 1$.

Also, since $\frac{q}{2} - 1$ is even $\prod_{u \in N(v)} f(u) = 1$for $v \in P$.

Now
$$|B| = |P^-| + |Q^-|$$
$$= \frac{p}{2} - 1 + \frac{q}{2} - 1 = \frac{p+q-4}{2}$$
$$|A| = p + q - \left[\frac{p+q-4}{2}\right] = \frac{p+q+4}{2}.$$
$$\gamma_{(st)^*}(K_{p,q}) = w(f) = |A| + |B| = 4.$$

Case 3: q is odd and $\frac{q-1}{2}$ odd.

Since q is odd$| Q^-| \leq \frac{q-1}{2}$.

Also, since $\frac{q-1}{2}$ is odd $\prod_{u \in N(v)} f(u) = -1$ if $| Q^-| = \frac{q-1}{2}$ for $v \in P$.

Thus $| Q^-| = \frac{q-1}{2} - 1$

Now
$$|B| = |P^-| + |Q^-|$$
$$= \frac{p}{2} - 1 + \frac{q-1}{2} - 1 = \frac{p+q-5}{2}$$

$$|A| = p + q - \left[\frac{p+q-5}{2}\right] = \frac{p+q+5}{2}.$$

$$\gamma_{(st)^*}(K_{p,q}) = w(f) = |A| + |B| = 5.$$

Case 4: q is even and $\frac{q}{2} - 1$ odd.

Since q is even$|Q^-| \leq \frac{q}{2} - 1$.

Also, since $\frac{q}{2} - 1$ *is* odd $\prod_{u \in N(v)} f(u) = -1$ if $|Q^-| = \frac{q}{2} - 1$ for $v \in P$.

Thus $|Q^-| = \frac{q}{2} - 2$

Now
$$|B| = |P^-| + |Q^-|$$
$$= \frac{p}{2} - 1 + \frac{q}{2} - 2 = \frac{p+q-6}{2}$$
$$|A| = p + q - \left[\frac{p+q-6}{2}\right] = \frac{p+q+6}{2}.$$
$$\gamma_{(st)^*}(K_{p,q}) = w(f) = |A| + |B| = 6.$$

Hence

$$\gamma_{(st)^*}(K_{p,q}) = \begin{cases} 3 \text{ if } q \text{ is odd and } \frac{q-1}{2} \text{ even} \\ 4 \text{ if } q \text{ is even and } \frac{q}{2} - 1 \text{ even} \\ 5 \text{ if } q \text{ is odd and } \frac{q-1}{2} \text{ odd} \\ 6 \text{ if } q \text{ is even and } \frac{q}{2} - 1 \text{ odd} \end{cases}.$$

3. Conclusion

In this paper, we investigated the signed product domination number $\gamma_{s^*}(G)$ and the signed total product domination number $\gamma_{(st)^*}(G)$ for various standard graph classes, including paths, cycles, complete graphs, bipartite graphs, and multipartite graphs. The findings contribute to the broader field of domination in graph theory by extending traditional domination concepts to signed and product-based variations. These results can serve as a foundation for future research into more complex graph structures.

References

1. Harary F. Graph Theory, Narosa Publishing House, New Delhi; 1969.
2. Teresa W. Haynes., Stephen T. Hedetniemi., Peter J. Slater. Fundamentals of Domination of Graphs. Marcel Dekker. INC; 1998.
3. Bohdan Zelinka. Signed Total Domination Number of a Graph. Czechoslovak Mathematical Journal. 2001; 51(2): 225 - 229.
4. Hongyu Liang. Signed and Minus Signed Domination in Complete Multipartite Graphs.Ars Combinatoria. 2012; 126: 133 - 142.
5. Jean E. Dunbar., Stephen T. Hedetniemi., Micheal A. Henning., Peter J. Slater. Signed Domination in Graphs. Graph Theory Combinatorics and Applications. 1995; 1: 311 - 322.